

Random Process - Spectral CharacteristicsPower Density Spectrum (or) Power spectral Density

A random variable varying with respect to time is a random process and if the variation is a known variation or deterministic variation, the corresponding process is a deterministic process or a signal. For a periodic signal, Fourier series is used for the study of its spectral behaviour.

If it is non-periodic, Fourier transform is used. It is given as $x(w) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$, where $x(t)$ is the deterministic process.

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$$\text{Let } x_T(t) = x(t) \text{ for } -T < t < T;$$

$$= 0 \quad \text{elsewhere.}$$

Now, $x_T(t)$ is of finite duration and we will apply Fourier Transformer to it

$$\therefore F(x_T(t)) = X_T(w) = \int_{-T}^T x_T(t) e^{-j\omega t} dt$$

Since, over $(-T, +T)$, $x_T(t) = x(t)$,

$$X_T(w) = \int_{-T}^T x(t) e^{-j\omega t} dt.$$

Energy of a signal is defined as $E = \int_{-\infty}^{\infty} x^2(t) dt$

Here, Energy of $x_T(t) = \int_{-T}^T x^2(t) dt$.

Consider the Parseval's theorem for energy signals.

$$E = \int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Using Inverse Fourier Transform,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$E = \int_{-\infty}^{\infty} x^2(t) dt = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right]^2 dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \left[\int_{-\infty}^{\infty} x^2(t) e^{-j\omega t} dt \right] d\omega$$

$$\int_{-T}^T x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Since, power is time average of energy, power available over the interval $(-T, T)$ is

$$\frac{1}{2T} \int_{-T}^T x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

This $\frac{|X_T(\omega)|^2}{2T}$ is referred to as power density spectrum.

$$P_{AV} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x^2(t)] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

$\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$ is named as power spectral density

$S_X(\omega)$, then the average power is given

$$\approx P_{AV} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

Auto correlation function (ACF) of a Random process: $R_{xx}(T)$

The auto-correlation Function $R_{xx}(t_1 t_2)$ of a random process $x(t)$ is defined as the expectation of the product of two random variables $x(t_1)$ and $x(t_2)$ i.e., the values of the same process $x(t)$ at two different times t_1 and t_2 .

$$R_{xx}(t_1 t_2) = \overline{x(t_1)x(t_2)} = E[x(t_1)x(t_2)] \\ = \int \int \alpha_1 \alpha_2 f_x(t_1) f_x(t_2) (\alpha_1, \alpha_2) d\alpha_1 d\alpha_2.$$

For stationary random process the ACF will depend on the time difference $t_2 - t_1$ only, rather than on the absolute times t_1 and t_2 .
Thus for stationary process

$$R_{xx}(t_1 t_2) = R_{xx}(t_2 - t_1) \\ = R_{xx}(t) \\ = E[x(t)x(t+T)] \\ = \int_{-\infty}^{\infty} \alpha(t) \alpha(t+T) f_x(t) x(t+T) dt.$$

H is also defined by the time average is

$$R_x(T) = \langle \alpha(t)\alpha(t+T) \rangle \\ = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \alpha(t)\alpha(t+T) dt.$$

Thus the ACF is a measure of similarity for the given samples of a random process.

Find the PSD of a random process.

$z(t) = x(t) + y(t)$ where $x(t)$ and $y(t)$ are zero mean, individual random process.

Sol:

$$\text{Consider } R_{zz}(t, t+T) = E[z(t)z(t+T)].$$

$$= E[\overline{x(t)} + \overline{y(t)}] \cdot [\overline{x(t+T)} + \overline{y(t+T)}]$$

$$= E[x(t)x(t+T) + x(t)y(t+T) + y(t)x(t+T) + y(t)y(t+T)].$$

$$= R_{xx}(T) + R_{xy}(T) + R_{yx}(T) + R_{yy}(T)$$

Since $x(t)$ & $y(t)$ are zero mean, independent random processes,

$$R_{xy}(T) = R_{yx}(T) = 0$$

$$R_{zz}(T) = R_{xx}(T) + R_{yy}(T).$$

Since $x(t)$ and $y(t)$ are zero mean, independent random processes

Take Fourier Transform on both sides

$$S_{zz}(w) = S_{xx}(w) + S_{yy}(w).$$

Properties of power spectral density:

- For a WSS process PSD at zero frequency gives the area under the graph of auto correlation of the process.

$$S_{xx}(w) = \int_{-\infty}^{\infty} R_{xx}(t) e^{-jwT} dt.$$

$\Rightarrow S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(t) dt$ which is the area under auto correlation function.

2. The mean square value of WSS process equals the area under the graph of power spectral density. 22

$$R_{xx}(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega T} d\omega.$$

$\Rightarrow R_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$. which is the area under PSD curve. But, for a WSS process, $R_{xx}(0) = E[X^2(t)]$.

$$\therefore E[X^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega.$$

Cross Power Density spectrum :-

let $x(t)$ and $y(t)$ be two random processes and let $x_T(t)$ and $y_T(t)$ be their sample functions respectively
 let $x_T(t) = x(t) \quad \text{for } -T < t < T.$
 $= 0 \quad \text{elsewhere}$

The individual Fourier transformer of $x_T(t)$ and $y_T(t)$ are

$$x_T(w) = \int_{-T}^T x_T(t) e^{-j\omega t} dt$$

$$= \int_{-T}^T x(t) e^{-j\omega t} dt$$

$$y_T(w) = \int_{-T}^T y_T(t) e^{-j\omega t} dt$$

$$= \int_{-T}^T y(t) e^{-j\omega t} dt$$

Consider the generalized Parseval's relation

$$\text{Consider } \int_{-\infty}^{\infty} x_T(t) y(t) dt$$

$$\text{we have } y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega.$$

$$\therefore \int_{-\infty}^{\infty} x(t) y(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} y(w) e^{j\omega t} dw \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} y(w) \left[\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right] dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(-w) y(w) dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x^*(w) y(w) dw$$

$$\therefore \int_{-\infty}^{\infty} x(t) y(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^*(w) y(w) dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(w) y^*(w) dw$$

The average power $P_{xy} = \frac{1}{2T} \int_{-T}^T x(t) y(t) dt$

$$= \frac{1}{2T} \int_{-T}^T x(t) y(t) dt$$

By using the above parseval's relation.

$$P_{xy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x^*(w) y(w)}{2\pi} dw.$$

Using the similar argument as made in the case of power spectral density.

$$P_{xy}(\text{av}) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[x(t)y(t)] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E[x_T^*(w) \cdot y_T(w)]}{2\pi} dw.$$

$$P_{xy \text{av}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(w) dw$$

Similarly $P_{xy \text{av}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(w) dw$

A random process has the power density spectrum $S_{XX}(\omega) = \frac{\omega^2}{1+\omega^2}$. Find the average power in the process.

$$S_{XX}(\omega) = \frac{\omega^2}{1+\omega^2} = \frac{1+\omega^2-1}{1+\omega^2} = 1 - \frac{1}{1+\omega^2}$$

$$S_{XX}(\omega) = 1 - \frac{1}{2} \cdot \frac{2}{1+\omega^2}$$

$$R_{XX}(T) = F^{-1}[S_{XX}(\omega)] = \delta(T) - \frac{1}{2} e^{-|T|}.$$

$$\text{Average power} = R_{XX}(0) = 1 - \frac{1}{2} = \frac{1}{2}.$$

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